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On the structure of the tight-span of a totally split-decomposable metric

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Abstract

The tight-span of a finite metric space is a polytopal complex with a structure that reflects properties of the metric. In this paper we consider the tight-span of a totally split-decomposable metric. Such metrics are used in the field of phylogenetic analysis, and a better knowledge of the structure of their tight-spans should ultimately provide improved phylogenetic techniques. Here we prove that a totally split-decomposable metric is cell-decomposable. This allows us to break up the tight-span of a totally split-decomposable metric into smaller, easier to understand tight-spans. As a consequence we prove that the cells in the tight-span of a totally split-decomposable metric are zonotopes that are polytope isomorphic to either hypercubes or rhombic dodecahedra.

1. Introduction

In this paper X will denote a finite set with $|X| \geq 2$. Given a metric d on X, i.e. a symmetric map $d : X \times X \to \mathbb{R}$ that vanishes precisely on the diagonal and satisfies the usual triangle inequality, associate a polytopal complex $T(d)$ to d as follows. Let $\mathbb{R}^X$

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denote the set of functions that map X to $\mathbb{R}$. Associate the polyhedron

$$P(d) = \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\},$$

to d, and let $T(d)$ consist of the bounded faces of $P(d)$. The complex $T(d)$ is known as the *tight-span* of d.

The tight-span of a metric was first introduced by Isbell [18], and has been subsequently rediscovered and studied in e.g. [3,7]. More recently, tight spans have been seen to arise naturally in the context of *tropical geometry*, where they have been shown to be related to *tropical polytopes* [4,5]. The structure of $T(d)$ can be used to deduce properties of d. For example, a study of metrics having "tree-like" tight-spans in [7] led in part to the concept of a *totally split-decomposable* metric [2], which, besides having applications to the theory of finite metric spaces [6], is now regularly used within phylogenetic analysis (cf. e.g. [1, 13,17,20]).

In this paper we are interested in better understanding the structure of the tight-span of a totally split-decomposable metric. Building upon results on this structure presented in [9,10,12], in our main result (Theorem 7.1) we prove that the tight-span of a totally split-decomposable metric is *cell-decomposable*, a property that implies that the tight-span consists of smaller, easier to understand tight-spans. As a consequence, using results in [15,16], we prove in Corollary 7.3 that the cells in the tight-span of a totally split-decomposable metric are zonotopes that are polytope isomorphic either to hypercubes or rhombic dodecahedra. We expect that this improved understanding of the tight-span will ultimately lead to better tools for phylogenetic analysis.

The rest of the paper is organised as follows. In Section 2, we present some preliminaries including a definition for the *Buneman complex*. This is a polyhedral complex that can be associated to a totally split-decomposable metric. In Section 3 we present some new results concerning the structure of the Buneman complex. We then consider a map κ introduced in [9], that relates the Buneman complex of a totally split-decomposable metric to its tight-span. In particular, in Theorem 4.3 we characterise when κ maps the Buneman complex into the tight-span. As a corollary, in Section 5 we give conditions for when κ induces an injection from the set of maximal cells of the Buneman complex into the set of the maximal cells of the tight-span, and subsequently, in Section 6, we characterise when κ induces a bijection. Using these results together with some from [15,16], in Section 7, we conclude with the proofs of Theorem 7.1 and Corollary 7.3.

2. Preliminaries

In this section, we review some properties of the Buneman complex and the tight-span. We begin by recalling some basic definitions concerning polytopes and polytopal complexes.

2.1. Polytopal complexes

We follow [19] and [21]. A *polyhedron* in $\mathbb{R}^n$, $n \in \mathbb{N}$, is the intersection of a finite collection of halfspaces in $\mathbb{R}^n$ and a *polytope* is a bounded polyhedron. A *face* of a polyhedron P is the empty-set, P itself, or the intersection of P with a supporting hyperplane and, if $\dim(P) = d$, i.e. P is d-dimensional, then its 0-dimensional faces are called its *vertices*. The collection of all faces of a polytope forms a lattice with respect to the ordering given by set inclusion, and we say that two polytopes are *polytope isomorphic* if their face-lattices are isomorphic. A *polyhedral complex* $\mathcal{C}$ is a finite collection of polyhedra (which we call *cells*) such that each face of a member of $\mathcal{C}$ is itself a member of $\mathcal{C}$, and the intersection of two members of $\mathcal{C}$ is a face of each. If all of the cells in $\mathcal{C}$ are polytopes, we call $\mathcal{C}$ a *polytopal complex*. Given a polyhedral complex $\mathcal{C}$, we will not usually distinguish between $\mathcal{C}$ and its *underlying set* $\bigcup_{C\in\mathcal{C}} C$. For any c in the underlying set of $\mathcal{C}$, we let $[c]$ denote the minimal cell in $\mathcal{C}$ (under inclusion of cells), that contains c. Also, if c is in the underlying set of $\mathcal{C}$ and C is a cell in $\mathcal{C}$ with $C = [c]$, then we say that c is a *generator* of C.

2.2. Totally split-decomposable metrics

A *split* of X is a bipartition of X, and a set $\mathcal{S}$ of splits of X is a *split system* (on X). Denote the split system consisting of all possible splits of X by $\mathcal{S}(X)$. For every $x \in X$ and any split S of X, we denote by $S(x)$ the element of S that contains x, and by $\overline{S}(x)$ the complement of $S(x)$. To avoid certain non-essential technicalities, in this paper we will assume that all split systems are non-empty and, for $\mathcal{S}$ a split system on X, that, for all $x \neq y$ in X, there exists some split $S \in \mathcal{S}$ with $S(x) \neq S(y)$. A *weighting* on a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is a map $\alpha : \mathcal{S} \to \mathbb{R}^{>0} : S \mapsto \alpha_S = \alpha(S)$, and such a pair $(\mathcal{S}, \alpha)$ is called a *weighted split system* (on X). We call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *weakly compatible* if there exist no three distinct splits $S_1, S_2, S_3 \in \mathcal{S}$ and four elements $x_0, x_1, x_2, x_3 \in X$ such that

$$S_j(x_i) = S_j(x_0) \text{ if and only if } i = j. \tag{1}$$

Now, a metric d on X is called *totally split-decomposable* if there exists a weighted weakly compatible split system $(\mathcal{S}, \alpha)$ on X with

$$d = d_{\mathcal{S},\alpha} = \sum_{S\in\mathcal{S}} \alpha_S \delta_S,$$

where, for any split $S \in \mathcal{S}(X)$ and all $x, y \in X$,

$$\delta_S(x, y) = \begin{cases} 1 & \text{if } S(x) \neq S(y), \\ 0 & \text{else.} \end{cases}$$

Note that if d is such a metric, then it follows by results in [2] that if $d = d_{\mathcal{S}',\alpha'}$ for some weakly compatible split system $\mathcal{S}'$ and weighting α' on $\mathcal{S}'$, then $\mathcal{S}' = \mathcal{S}$ and $\alpha' = \alpha$. Totally split-decomposable metrics were introduced in [2]. Besides having mathematical interest, such metrics play a useful role in phylogenetic analysis (cf. e.g. [13,17]).

2.3. The Buneman complex

We begin by recalling some further definitions concerning splits and split systems. Recall that X is a finite set. For every proper non-empty subset $A \subseteq X$, we denote the split $\{A, \overline{A}\}$ by S_A. Given a split system $\mathcal{S} \subseteq \mathcal{S}(X)$, we define its underlying set $\mathcal{U}(\mathcal{S})$ by

$$\mathcal{U}(\mathcal{S}) = \bigcup_{S\in\mathcal{S}} S = \{A \subseteq X \mid \text{ there exists } S \in \mathcal{S} \text{ with } A \in S\}.$$

We call two distinct splits $S, S' \in \mathcal{S}(X)$ *compatible* if there exists some $A \in S$ and some $A' \in S'$ with $A \cap A' = \emptyset$, otherwise we call S and S' *incompatible*. We call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *incompatible* if every pair of distinct splits in $\mathcal{S}$ is incompatible. We also define any split system with cardinality one to be incompatible.

Now, given any map $\phi : \mathcal{U}(\mathcal{S}) \to \mathbb{R}$, we define

$$supp(\phi) = \{A \in \mathcal{U}(\mathcal{S}) \mid \phi(A) \neq 0\},$$

and put

$$\mathcal{S}(\phi) = \{S \in \mathcal{S} : S \subseteq supp(\phi)\}.$$

Given a weighted split system $(\mathcal{S}, \alpha)$ on X, put

$$H(\mathcal{S}, \alpha) = \left\{\phi \in \mathbb{R}^{\mathcal{U}(\mathcal{S})} : \phi(A) \geq 0 \text{ and } \phi(A) + \phi(\overline{A}) = \frac{\alpha_{S_A}}{2} \text{ for all } A \in \mathcal{U}(\mathcal{S})\right\}.$$

It is straight-forward to check that this is a polytope in $\mathbb{R}^{\mathcal{U}(\mathcal{S})}$ that is polytope isomorphic to an $|\mathcal{S}|$-dimensional hypercube. The subset $B(\mathcal{S}, \alpha)$ of $H(\mathcal{S}, \alpha)$ defined by

$$B(\mathcal{S}, \alpha) = \{\phi \in H(\mathcal{S}, \alpha) : A_1, A_2 \in supp(\phi) \text{ and } A_1 \cup A_2 = X \Rightarrow A_1 \cap A_2 = \emptyset\}$$

is a polytopal complex called the *Buneman complex* associated to $(\mathcal{S}, \alpha)$. This complex was introduced in [8]—see also [9] (note in the definition that we present for $H(\mathcal{S}, \alpha)$, we have introduced a factor of $\frac{1}{2}$ for scaling purposes).

It can be shown that the map $d_1 : \mathbb{R}^{\mathcal{U}(\mathcal{S})} \times \mathbb{R}^{\mathcal{U}(\mathcal{S})} \to \mathbb{R}^{\geq 0}$ defined, for all $\phi, \phi' \in \mathbb{R}^{\mathcal{U}(\mathcal{S})}$, by

$$d_1(\phi, \phi') = \sum_{A \in \mathcal{U}(\mathcal{S})} |\phi(A) - \phi'(A)|$$

restricts to give a metric on both $H(\mathcal{S}, \alpha)$ and $B(\mathcal{S}, \alpha)$, and that the map from X into $B(\mathcal{S}, \alpha)$ defined by taking an element $x \in X$ to the function

$$\phi_x : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \frac{\alpha_{S_A}}{2} & \text{if } x \notin A, \\ 0 & \text{else,} \end{cases}$$

is an embedding of $(X, d_{\mathcal{S},\alpha})$ into $(B(\mathcal{S}, \alpha), d_1)$ [8, Section 2]. We will make use of the following results:

(B1) [8, Section 2] If $\phi \in B(\mathcal{S}, \alpha)$, then

$$[\phi] = \{\psi \in H(\mathcal{S}, \alpha) \mid supp(\psi) \subseteq supp(\phi)\}.$$

(B2) [8, Lemma 5.2] For all $\mathcal{S}' \subseteq \mathcal{S}$ the (restriction) map

$$B(\mathcal{S}, \alpha) \to B(\mathcal{S}', \alpha) : \phi \mapsto \phi|_{\mathcal{S}'}$$

is surjective.

(B3) $\mathcal{S}' \subseteq \mathcal{S}$ is a maximal incompatible split system, then there exists a unique maximal cell C in $B(\mathcal{S}, \alpha)$ with $\mathcal{S}(\phi) = \mathcal{S}'$, for any generator ϕ of C. (This follows by [8, Proposition 3.3], which states that a split system $\mathcal{S}$ is incompatible if and only if $H(\mathcal{S}, \alpha) = B(\mathcal{S}, \alpha)$, and (B2).)

(B4) If $\phi \in B(\mathcal{S}, \alpha)$ with $[\phi]$ a maximal cell of $B(\mathcal{S}, \alpha)$, then $\mathcal{S}(\phi)$ is a maximal incompatible split system in $\mathcal{S}$. (This follows from the definition of $B(\mathcal{S}, \alpha)$, (B1), (B2), and [8, Proposition 3.3].)

(B5) [8, Section 2] If C is a cell in $B(\mathcal{S}, \alpha)$ and ϕ is any generator of C, then $\dim(C) = |\mathcal{S}(\phi)|$.

2.4. The tight-span

Suppose that d is a metric on X. It can be shown (cf. [7,18]) that the map $d_\infty : \mathbb{R}^X \times \mathbb{R}^X \to \mathbb{R}^{\geq 0}$ defined, for $f, g \in \mathbb{R}^X$, by

$$d_\infty(f, g) = \max_{x \in X} |f(x) - g(x)|,$$

restricts to give a metric on $P(d)$ and $T(d)$, and that the map

$$\Psi : X \to T(d) : y \mapsto (h_y : X \to \mathbb{R} : x \mapsto d(x, y))$$

is an embedding of the metric space (X, d) into $(T(d), d_\infty)$, i.e. Ψ is an injection with $d_\infty(\Psi(x), \Psi(y)) = d(x, y)$ holding for all $x, y \in X$.

Now, given $f \in P(d)$, define a graph $K(f)$ with vertex set X and edge set consisting of those subsets $\{x, y\}$ of X with $f(x) + f(y) = d(x, y)$. Proofs for the following statements can be found in [7]:

(TS1) If $f \in T(d)$, then

$$[f] = \{g \in T(d) : K(f) \subseteq K(g)\}.$$

(TS2) If $f \in P(d)$, then $f \in T(d)$ if and only if for all $x \in X$ there is some $y \in X$ distinct from x with $\{x, y\}$ an edge of $K(f)$.

(TS3) If $f \in T(d)$ and $f(y) = 0$ for some $y \in X$, then $f = h_y$.

3. Gates in the Buneman complex

In this section, we prove some results concerning the Buneman complex. Suppose that $(\mathcal{S}, \alpha)$ is any weighted split system on X. Note that every cell C in the Buneman complex $B(\mathcal{S}, \alpha)$ is X*-gated*, that is, for every $x \in X$ there is an element γ in C, called the *gate for* x *in* C, with

$$d_1(\phi_x, \psi) = d_1(\phi_x, \gamma) + d_1(\gamma, \psi)$$

holding for all $\psi \in C$. Now, for any $x \in X$ and any generator ϕ of C, define

$$\gamma^x = \gamma^x_C : \mathcal{U}(\mathcal{S}) \to \mathbb{R}^{\geq 0} : A \mapsto \begin{cases} \phi_x(A) & \text{if } A \in \mathcal{U}(\mathcal{S}(\phi)), \\ \phi(A) & \text{else.} \end{cases}$$

We first prove that γ^x is a gate for x in C.

Lemma 3.1. *Suppose $(\mathcal{S}, \alpha)$ is a weighted split system on X, C is any cell in $B(\mathcal{S}, \alpha)$, and $\phi \in B(\mathcal{S}, \alpha)$ is any generator of C. Then the following statements hold.*

(i) *If $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$ and $\psi \in C$, then $\phi(A) \in \{0, \frac{\alpha_{S_A}}{2}\}$ and $\psi(A) = \phi(A)$.*

(ii) *For any* $x \in X$, *the map* γ^x *defined above is a gate for* x *in* C.

(iii) *For all* $x, y \in X$,

$$d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S \delta_S(x, y).$$

Proof. Suppose $x, y \in X$, C is a cell in $B(\mathcal{S}, \alpha)$, and $\phi \in B(\mathcal{S}, \alpha)$ is a generator of C. It is straight-forward to see that (i)–(iii) all hold in case C is a vertex. So, without loss of generality, we will assume $\dim(C) > 0$. In particular, by (B5) $\mathcal{S}(\phi) \neq \emptyset$.

(i): Suppose $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. Then $|\{A, \overline{A}\} \cap supp(\phi)| = 1$, by the definition of $\mathcal{S}(\phi)$. Hence $\phi(A) \in \{0, \frac{\alpha_{S_A}}{2}\}$, as $\phi \in H(\mathcal{S}, \alpha)$. Now suppose $\psi \in C$. By (B1) it follows that $|\{A, \overline{A}\} \cap supp(\psi)| \leq |\{A, \overline{A}\} \cap supp(\phi)| = 1$ and so, again by (B1), $\psi(A) = \phi(A)$.

(ii): By (B1) and the definition of γ^x, ϕ_x, and $\mathcal{S}(\phi)$, it follows that γ^x is contained in C. Now suppose ψ is any element of C. By (i) and the definition of γ^x,

$$\begin{aligned} d_1(\phi_x, \psi) &= \sum_{A \in \mathcal{U}(\mathcal{S})} |\phi_x(A) - \psi(A)| \\ &= \sum_{A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))} |\phi_x(A) - \psi(A)| + \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A) - \psi(A)| \\ &= \sum_{A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))} |\phi_x(A) - \gamma^x(A)| + \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\gamma^x(A) - \psi(A)| \\ &= d_1(\phi_x, \gamma^x) + d_1(\gamma^x, \psi). \end{aligned}$$

Hence γ^x is a gate for x in C.

(iii): By (i), (ii), and the definition of ϕ_z and γ^z, $z \in X$,

$$\begin{aligned} d_1(\gamma^x, \gamma^y) &= \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A) - \phi_y(A)| \\ &= \sum_{S \in \mathcal{S}(\phi)} \phi_y(S(x)) + \sum_{S \in \mathcal{S}(\phi)} \left| \frac{\alpha_S}{2} - \phi_y(\overline{S}(x)) \right| \\ &= \sum_{S \in \mathcal{S}(\phi)} \alpha_S \delta_S(x, y). \quad \square \end{aligned}$$

Note that if C is a cell of $B(\mathcal{S}, \alpha)$ with $\dim(C) > 0$, then d_1 restricted to the set

$$\Gamma(C) = \{\gamma_C^x : x \in X\}$$

is a metric. Later we shall be interested in the case where the metric $d_1|_{\Gamma(C)}$ is *antipodal* (recall that a metric d on a finite set Y is antipodal if there is an involution $\sigma : Y \to Y$, mapping each element y in Y to an element $\overline{y}$ distinct from y, called the *antipode* of y, with $d(y, z) + d(z, \overline{y}) = d(y, \overline{y})$ holding for all $z \in Y$). In this situation, we call C *antipodal* X*-gated* and, for $x, y \in X$, we call γ^x the *antipode* of γ^y in C if γ^x is the antipode of γ^y in $(\Gamma(C), d_1|_{\Gamma(C)})$.

Proposition 3.2. *Suppose* $(\mathcal{S}, \alpha)$ *is a weighted split system on* X, C *is a cell in* $B(\mathcal{S}, \alpha)$ *with* $\dim(C) > 0$, *and* ϕ *is any generator of* C. *Suppose in addition that* $d_1|_{\Gamma(C)}$ *is* *an antipodal metric on $\Gamma(C)$, and $x, y \in X$ distinct. Then the following statements are equivalent.*

(i) *γ^x is the antipode of γ^y in C.*

(ii) *$S(x) \neq S(y)$ for all $S \in \mathcal{S}(\phi)$.*

(iii) *$d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \psi) + d_1(\psi, \gamma^y)$ for all $\psi \in C$.*

(iv) *For all $S \in \mathcal{S}(\phi)$ and all $\psi \in C$,*

$$\alpha_S = \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|.$$

(v) *$d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S$.*

Proof. (i) $\Rightarrow$ (ii): Suppose γ^x is the antipode of γ^y in C, and that there is some $S_0 \in \mathcal{S}(\phi)$ with $S_0(x) = S_0(y)$. Let $z \in \overline{S_0}(x)$. Since γ^x is the antipode of γ^y,

$$d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \gamma^z) + d_1(\gamma^z, \gamma^y),$$

and so, using Lemma 3.1, $\alpha_S \delta_S(x, y) = \sum_{A \in S} |\phi_x(A) - \phi_z(A)| + |\phi_z(A) - \phi_y(A)|$, for all $S \in \mathcal{S}(\phi)$. Thus,

$$\begin{aligned} 0 &= \alpha_{S_0} \delta_{S_0}(x, y) \\ &= \sum_{A \in S_0} |\phi_x(A) - \phi_z(A)| + |\phi_z(A) - \phi_y(A)| \\ &= 2(|\phi_x(S_0(x)) - \phi_z(S_0(x))| + |\phi_z(S_0(x)) - \phi_y(S_0(x))|) \\ &= 4\phi_z(S_0(x)), \end{aligned}$$

and so $\phi_z(S_0(x)) = 0$. Thus $z \in S_0(x)$, a contradiction.

(ii) $\Rightarrow$ (iv): Suppose $S \in \mathcal{S}(\phi)$ and $\psi \in C$. Then

$$\begin{aligned} \alpha_S &= \psi(S(x)) + \frac{\alpha_S}{2} - \psi(S(x)) + \frac{\alpha_S}{2} - \psi(\overline{S}(x)) + \psi(\overline{S}(x)) \\ &= \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\phi_y(A) - \psi(A)|. \end{aligned}$$

(iv) $\Rightarrow$ (iii): Suppose $\psi \in C$. By Lemma 3.1(i), $\gamma^x(A) = \gamma^y(A) = \phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. (iii) now follows.

(iii) $\Rightarrow$ (i): This is clear since $\Gamma(C) \subseteq C$.

(ii) $\Rightarrow$ (v): This follows by Lemma 3.1(iii).

(v) $\Rightarrow$ (ii): Suppose (v) holds and there exists some $S' \in \mathcal{S}(\phi)$ with $S'(x) = S'(y)$. Then since $\sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\phi_x(A) - \phi_y(A)| = d_1(\gamma^x, \gamma^y) = \sum_{S \in \mathcal{S}(\phi)} \alpha_S$, there must exist some $S'' \in \mathcal{S}(\phi)$ with

$$\alpha_{S''} < \sum_{A \in S''} |\phi_x(A) - \phi_y(A)| = 2|\phi_x(S''(x)) - \phi_y(S''(x))| = \alpha_{S''},$$

which is impossible. $\square$

Corollary 3.3. *Suppose that the conditions stated in the last proposition all hold and that in addition the cell C is maximal. Then the following statements hold.*

(i) *If $S(x) \neq S(y)$ for all $S \in \mathcal{S}(\phi)$, then $\phi(S'(x)) = 0$ for all $S' \in \mathcal{S} - \mathcal{S}(\phi)$ with $S'(x) = S'(y)$.*
(ii) *$\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$ if and only if γ^x is the antipode of γ^y in C.*
(iii) *Suppose $x_1, x_2, y_1, y_2 \in X$ and that the antipode of γ^{y_i} in C is γ^{x_j} for all $i, j \in \{1, 2\}$. Then $d_{\mathcal{S},\alpha}(x_1, y_1) + d_{\mathcal{S},\alpha}(x_2, y_2) = d_{\mathcal{S},\alpha}(x_1, y_2) + d_{\mathcal{S},\alpha}(x_2, y_1)$.*

Proof. (i): Suppose $S' \in \mathcal{S} - \mathcal{S}(\phi)$ with $S'(x) = S'(y)$. Since $\mathcal{S}(\phi)$ is maximal incompatible by (B4), there must exist some $S \in \mathcal{S}(\phi)$ which is compatible with S'. As $y \in S'(x)$ and $S(x) \neq S(y)$ by assumption, either $S(x) \cup S'(x) = X$ or $S(y) \cup S'(x) = X$. Since $S(x), S(y) \in supp(\phi)$ and $S(x) \cap S'(x) \neq \emptyset \neq S(y) \cap S'(y) = S(y) \cap S'(x)$, it follows that $\phi(S'(x)) = 0$.
(ii): Suppose $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$. Then clearly $d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \phi) + d_1(\phi, \gamma^y)$. Hence, by Proposition 3.2, γ^x is the antipode of γ^y in C.

Conversely, suppose that γ^x is the antipode of γ^y in C. By (i)

$$\alpha_S \delta_S(x, y) = 2(\phi(S(x)) + |\phi_y(S(x)) - \phi(S(x))|)$$

for all $S \in \mathcal{S} - \mathcal{S}(\phi)$. Now using Lemma 3.1 and Proposition 3.2, it is straight-forward to check that

$$\begin{aligned}
d_1(\phi_x, \phi_y) &= \sum_{S \in \mathcal{S}} \alpha_S \delta_S(x, y) \\
&= \sum_{S \in \mathcal{S}(\phi)} \alpha_S + \sum_{S \in \mathcal{S} - \mathcal{S}(\phi)} \alpha_S \delta_S(x, y) \\
&= d_1(\gamma^x, \gamma^y) + \sum_{S \in \mathcal{S} - \mathcal{S}(\phi)} 2(\phi(S(x)) + |\phi_y(S(x)) - \phi(S(x))|) \\
&= d_1(\gamma^x, \gamma^y) + \sum_{A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))} (|\phi_x(A) - \phi(A)| + |\phi_y(A) - \phi(A)|) \\
&= d_1(\gamma^x, \gamma^y) + d_1(\phi_x, \gamma^x) + d_1(\gamma^y, \phi_y)
\end{aligned}$$

holds. But by Proposition 3.2, $d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \phi) + d_1(\phi, \gamma^y)$. It immediately follows that $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S}, \alpha)$.
(iii): Using (i) and Proposition 3.2 it is straight-forward to show that

$$d_{\mathcal{S},\alpha}(x_i, y_j) = d_1(\phi_{x_i}, \gamma^{x_i}) + d_1(\gamma^{x_i}, \gamma^{y_j}) + d_1(\gamma^{y_j}, \phi_{y_j})$$

holds for all $i, j \in \{1, 2\}$. But by uniqueness of gates, $\gamma^{x_1} = \gamma^{x_2}$ and $\gamma^{y_1} = \gamma^{y_2}$ and (iii) now easily follows. □

4. Teutoburgan split systems

Given a weighted split system $(\mathcal{S}, \alpha)$ on X, define a map

$$\kappa : \mathbb{R}^{\mathcal{U}(\mathcal{S})} \to \mathbb{R}^X : \phi \mapsto (X \to \mathbb{R} : x \mapsto d_1(\phi, \phi_x)).$$

The map κ was originally introduced in [9].[1] Note that it immediately follows from this definition that $\kappa(B(\mathcal{S},\alpha)) \subseteq P(d_{\mathcal{S},\alpha})$ and that, by (TS3), $\kappa(\phi_x) = h_x$, for all $x \in X$. Moreover, using the fact that, for y a fixed element in X, and for all $x \in X$ and all $\phi \in B(\mathcal{S},\alpha)$,

$$\kappa(\phi)(x) = 2 \sum_{\{A \in \mathcal{U}(\mathcal{S}) : y \in A\}} |\phi(A) - \phi_x(A)|, \tag{2}$$

it is straight-forward to check that κ induces a non-expanding map from $B(\mathcal{S},\alpha)$ to $P(d_{\mathcal{S},\alpha})$, i.e.

$$d_\infty(\kappa(\phi), \kappa(\psi)) \leq d_1(\phi, \psi)$$

holds for all $\phi, \psi \in B(\mathcal{S},\alpha)$. In this section we will characterise those split systems $\mathcal{S}$ of X for which $\kappa(B(\mathcal{S},\alpha)) \subseteq T(d_{\mathcal{S},\alpha})$ holds for any weighting α on $\mathcal{S}$.

We begin by proving two useful lemmas. Abusing notation, to any $\phi \in B(\mathcal{S},\alpha)$ associate the graph $K(\phi)$ which has vertex set X and edge set consisting of those subsets $\{x, y\}$ of X with $d_1(\phi_x, \phi_y) = d_1(\phi_x, \phi) + d_1(\phi, \phi_y)$. It is straight-forward to check that $\{x, y\}$ is an edge of $K(\phi)$ if and only if $\{x, y\}$ is an edge of $K(\kappa(\phi))$.

Lemma 4.1. *Let $(\mathcal{S},\alpha)$ be a weighted split system on X. Suppose C is a maximal cell in $B(\mathcal{S},\alpha)$, ϕ is any generator of C, and $d_1|_{\Gamma(C)}$ is an antipodal metric on $\Gamma(C)$. Then, for $x, y \in X$ distinct, the following statements are equivalent.*

(i) *γ^x is the antipode of γ^y in C.*
(ii) *$\{x, y\} \subseteq X$ is an edge of $K(\kappa(\phi))$ or – equivalently – of $K(\phi)$.*
(iii) *$\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$.*

Proof. (i) $\Rightarrow$ (iii): Suppose γ^x is the antipode of γ^y in C. By Corollary 3.3(ii), $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S},\alpha)$. Since κ is non-expanding, it immediately follows that $\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$.
(iii) $\Rightarrow$ (ii): Suppose $\kappa(\phi_x), \kappa(\gamma^x), \kappa(\phi), \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $P(d_{\mathcal{S},\alpha})$. Then clearly

$$d_\infty(\kappa(\phi_x), \kappa(\phi)) + d_\infty(\kappa(\phi), \kappa(\phi_y)) = d_\infty(\kappa(\phi_x), \kappa(\phi_y)).$$

But, for any $z \in X$, $\kappa(\phi_z) = h_z$ and so $d_\infty(\kappa(\phi_z), \kappa(\phi)) = d_\infty(h_z, \kappa(\phi)) = \kappa(\phi)(z)$. (ii) now follows immediately.
(ii) $\Rightarrow$ (i): Suppose $\{x, y\}$ is an edge of $K(\kappa(\phi))$. Then $d_1(\phi_x, \phi_y) = d_1(\phi_x, \phi) + d_1(\phi, \phi_y)$. Since γ^x and γ^y are gates in C for x and y, respectively, it immediately follows that $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ is a geodesic in $B(\mathcal{S},\alpha)$. Hence, by Corollary 3.3(ii), γ^x is the antipode of γ^y in C. $\square$

A split system $\mathcal{S} \subseteq \mathcal{S}(X)$ is called *antipodal* if for all $x \in X$ there exists some $y \in X$ such that $S(x) \neq S(y)$ holds for all $S \in \mathcal{S}$. Such split systems were studied in [11]. We now relate them to antipodal X-gated cells in the Buneman complex.

[1] In [9] this map is denoted by λ. Since our definition of κ is slightly different from the map λ presented in [9], we use κ as opposed to λ to prevent confusion. It can be easily checked that the results stated in [9] concerning λ also hold for κ.

Lemma 4.2. *Suppose $C \subseteq B(\mathcal{S}, \alpha)$ is a cell with $\dim(C) > 0$ and ϕ is a generator of C. Then the following statements are equivalent.*

(i) *C is antipodal X-gated.*
(ii) *$\mathcal{S}(\phi)$ is antipodal.*

Proof. (i) $\Rightarrow$ (ii): This follows immediately from Proposition 3.2.

(ii) $\Rightarrow$ (i): Suppose $x \in X$. Since $\mathcal{S}(\phi)$ is antipodal by assumption, there is some $y \in X$ with $S(x) \neq S(y)$ holding for all $S \in \mathcal{S}(\phi)$. Note that if $y' \in X$ distinct from y with $S(x) \neq S(y')$ for all $S \in \mathcal{S}(\phi)$ then $S(y) = S(y')$ and so $\gamma^y = \gamma^{y'}$ follows by the definition of γ^y and $\gamma^{y'}$. Hence, the map which takes, for any $u \in X$, the gate γ^u to γ^v with $v \in \bigcap_{S \in \mathcal{S}(\phi)} \overline{S}(u)$ is a well-defined involution on $\Gamma(C)$. Moreover, for all $z \in X$ and all $A \in \mathcal{U}(\mathcal{S}(\phi))$,

$$|\phi_x(A) - \phi_y(A)| = |\phi_x(A) - \phi_z(A)| - |\phi_z(A) - \phi_y(A)|$$

and hence, by Lemma 3.1(i),

$$d_1(\gamma^x, \gamma^y) = d_1(\gamma^x, \gamma^z) + d_1(\gamma^z, \gamma^y).$$

Thus $d_1|_{\Gamma(C)}$ is an antipodal metric on $\Gamma(C)$, and, therefore, C is antipodal X-gated. $\square$

We now give the characterisation promised above.

Theorem 4.3. *Suppose that $\mathcal{S}$ is a split system on X. Then the following statements are equivalent:*

(i) *Every maximal incompatible split system contained in $\mathcal{S}$ is antipodal.*
(ii) *For every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, every maximal cell in $B(\mathcal{S}, \alpha)$ is antipodal X-gated.*
(ii′) *For some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, every maximal cell in $B(\mathcal{S}, \alpha)$ is antipodal X-gated.*
(iii) *For every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, every cell in $B(\mathcal{S}, \alpha)$ with non-zero dimension is antipodal X-gated.*
(iii′) *For some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, every cell in $B(\mathcal{S}, \alpha)$ with non-zero dimension is antipodal X-gated.*
(iv) *For every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$.*
(iv′) *For some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$.*

Proof. We will prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii′) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iii′) $\Rightarrow$ (ii′), and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iv′) $\Rightarrow$ (ii′).

The implications (ii) $\Rightarrow$ (ii′), (iii) $\Rightarrow$ (iii′), (iv) $\Rightarrow$ (iv′), and (iii′) $\Rightarrow$ (ii′) clearly all hold.

(i) $\Rightarrow$ (ii): Suppose that $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, and C is a maximal cell in $B(\mathcal{S}, \alpha)$ with generator ϕ. By (B4), $\mathcal{S}(\phi)$ is maximal incompatible and so $\mathcal{S}(\phi)$ must be antipodal, by assumption. Thus, by Lemma 4.2, C is antipodal X-gated.

(ii) $\Rightarrow$ (iii): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a cell in $B(\mathcal{S}, \alpha)$ with $\dim(C) > 0$, and D is any maximal cell containing C. Let ϕ and ψ be generators of C and D, respectively. By assumption, D is antipodal X-gated, and so for any $x \in X$ there exists

some $y \in X$ with γ_D^y the antipode of γ_D^x in D. By Proposition 3.2, $S(x) \neq S(y)$ for all $S \in \mathcal{S}(\psi)$. By (B1), $\mathcal{S}(\phi) \subseteq \mathcal{S}(\psi)$ and so $\mathcal{S}(\phi)$ is antipodal. (iii) now follows by Lemma 4.2.
(ii′) ⇒ (i): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting so that every maximal cell in $B(\mathcal{S}, \alpha)$ is antipodal X-gated. Suppose $\mathcal{S}' \subseteq \mathcal{S}$ is a maximal incompatible split system. Then, by (B3), $\mathcal{S}' = \mathcal{S}(\phi)$ where $\phi \in B(\mathcal{S}, \alpha)$ is a generator of some maximal cell in $B(\mathcal{S}, \alpha)$. Since $[\phi]$ is antipodal X-gated by assumption, $\mathcal{S}'$ is antipodal by Lemma 4.2.
(ii) ⇒ (iv): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a maximal cell in $B(\mathcal{S}, \alpha)$, ϕ is a generator of C, and $x \in X$. Then there must exist some $y \in X$ distinct from x with γ^y the antipode of γ^x in C. By Lemma 4.1, $\{x, y\}$ is an edge of $K(\kappa(\phi))$. Since $\kappa(\phi) \in P(d_{\mathcal{S},\alpha})$, (TS2) implies $\kappa(\phi) \in T(d_{\mathcal{S},\alpha})$. By (TS1), it follows that $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$.
(iv′) ⇒ (ii′): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting with $\kappa(B(\mathcal{S}, \alpha)) \subseteq T(d_{\mathcal{S},\alpha})$. Let C be a maximal cell in $B(\mathcal{S}, \alpha)$ with generator ϕ, and let $x \in X$. Since $\kappa(\phi) \in T(d_{\mathcal{S},\alpha})$, by (TS2) there is some $y \in X$ distinct from x with $\{x, y\}$ an edge of $K(\kappa(\phi))$. Hence, for all $S \in \mathcal{S}(\phi)$, we must have $S(x) \neq S(y)$ since, otherwise, if there were some $S \in \mathcal{S}(\phi)$ with $S(x) = S(y)$ then

$$0 = \alpha_S \delta_S(x, y) = \sum_{A \in S} |\phi_x(A) - \phi(A)| + |\phi_y(A) - \phi(A)| = 4\phi(S(x)),$$

which is impossible since $S \in \mathcal{S}(\phi)$. Thus $\mathcal{S}(\phi)$ is antipodal, and so, by Lemma 4.2, C is antipodal X-gated. □

Motivated by this last theorem, we call a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ *Teutoburgan* if every maximal incompatible subset of splits in $\mathcal{S}$ is antipodal. Since every weakly compatible, yet incompatible split system is antipodal [11], it immediately follows that every weakly compatible split system is Teutoburgan. Note, however, that a Teutoburgan split system is not necessarily weakly compatible (e.g. take the split system of cardinality 3 on the set of vertices of a 3-cube induced by removing collections of parallel edges).

Remark 4.4. If $(\mathcal{S}, \alpha)$ is a weighted split system for which the map $\Phi : X \to B(\mathcal{S}, \alpha)$ maps X surjectively onto the set of vertices of $B(\mathcal{S}, \alpha)$, then it is straight-forward to check that $\mathcal{S}$ is Teutoburgan. Moreover, it can be shown that such a split system can be associated to any *Buneman graph* [8] (by taking X to be the vertex set of the graph, and $\mathcal{S}$ to be the split system induced by the “parallel classes” of edges of the Buneman graph). This provides a large additional class of Teutoburgan split systems.

5. Maximal cells of the tight-span

In this section we shall show that if $\mathcal{S}$ is a Teutoburgan split system then, for any weighting α on $\mathcal{S}$, κ induces an injective map from the set of maximal cells of $B(\mathcal{S}, \alpha)$ into the set of maximal cells of $T(d_{\mathcal{S},\alpha})$. This is essentially a consequence of the following result.

Theorem 5.1. *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X with $\mathcal{S}$ Teutoburgan. Suppose C is a maximal cell of $B(\mathcal{S}, \alpha)$, and ϕ is any generator of C. Then the following statements hold.*

(i) *For all* $x \in X$, $\kappa(\gamma^x) \in [\kappa(\phi)]$.
(ii) $\kappa(\phi)$ *is a generator of a maximal cell of* $T(d_{\mathcal{S},\alpha})$.
(iii) *Suppose* $\psi \in B(\mathcal{S}, \alpha)$. *Then* $\psi \in C$ *if and only if* $\kappa(\psi) \in [\kappa(\phi)]$.

Proof. (i): Suppose $\{u, v\}$ is an edge of $K(\kappa(\phi))$ with $u \neq v$, which exists by (TS2). Since $\mathcal{S}$ is Teutoburgan, by Lemma 4.1 γ^u is the antipode of γ^v in C. By Proposition 3.2 and Corollary 3.3(ii), $\phi_u, \gamma^u, \gamma^z, \gamma^v, \phi_v$ is a geodesic in $B(\mathcal{S}, \alpha)$. Hence, since γ^u and γ^v are gates in C for u and v, respectively,

$$d_1(\phi_u, \phi_v) = d_1(\phi_u, \gamma^z) + d_1(\gamma^z, \phi_v) = \kappa(\gamma^z)(u) + \kappa(\gamma^z)(v).$$

Hence $\{u, v\}$ is an edge of $K(\kappa(\gamma^z))$. Thus, $K(\kappa(\phi)) \subseteq K(\kappa(\gamma^z))$, and so by (TS1) $\kappa(\gamma^z) \in [\kappa(\phi)]$.

(ii): By Theorem 4.3 $[\kappa(\phi)]$ is a cell of $T(d_{\mathcal{S},\alpha})$. Suppose that $[\kappa(\phi)]$ is not maximal. Then there exists some $f \in T(d_{\mathcal{S},\alpha})$ with $[\kappa(\phi)] \subsetneq [f]$. By (TS1), $K(f) \subsetneq K(\kappa(\phi))$ and so there exist $x_1, y_1 \in X$ with $\{x_1, y_1\}$ an edge of $K(\kappa(\phi))$ but not of $K(f)$. Note that $x_1 \neq y_1$. For, if not, then $\kappa(\phi) = h_{x_1}$ by (TS3), and, taking γ^z to be the antipode of γ^{x_1} in C, for $z \in X$ (which exists by Theorem 4.3), by (i) we obtain $\kappa(\gamma^z) = \kappa(\phi) = h_{x_1}$. So $\kappa(\gamma^z)(x_1) = 0$, which is impossible because, since γ^z is the antipode of γ^{x_1} in C,

$$d_1(\gamma^z, \phi_{x_1}) \geq \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\gamma^z(A) - \phi_{x_1}(A)| = \sum_{A \in \mathcal{U}(\mathcal{S}(\phi))} |\gamma^z(A) - \gamma^{x_1}(A)| > 0.$$

Now define

$$\begin{aligned} Z &= \{z \in X \mid \gamma_C^z = \gamma_C^{y_1}\}, \text{ and} \\ Y &= \{z \in X \mid \gamma_C^z \text{ is the antipode of } \gamma_C^{y_1} \text{ in } C\}. \end{aligned}$$

Clearly, $y_1 \in Z$ and, by Lemma 4.1, $x_1 \in Y$. Since $f \in T(d_{\mathcal{S},\alpha})$ and $\{x_1, y_1\}$ is not an edge of $K(f)$, by (TS2) there exist $x_2, y_2 \in X$ with $x_1 \neq y_2$ and $x_2 \neq y_1$ such that $\{x_1, y_2\}$ and $\{x_2, y_1\}$ are edges of $K(f)$. Since $K(f) \subseteq K(\kappa(\phi))$, Lemma 4.1 implies $x_2 \in Y$ and $y_2 \in Z$. Hence, by Corollary 3.3(iii), $d_{\mathcal{S},\alpha}(x_1, y_1) + d_{\mathcal{S},\alpha}(x_2, y_2) = d_{\mathcal{S},\alpha}(x_1, y_2) + d_{\mathcal{S},\alpha}(x_2, y_1)$. But then

$$\begin{aligned} f(y_1) + f(x_2) + f(y_2) + f(x_1) &= d_{\mathcal{S},\alpha}(y_1, x_2) + d_{\mathcal{S},\alpha}(y_2, x_1) \\ &= d_{\mathcal{S},\alpha}(y_1, x_1) + d_{\mathcal{S},\alpha}(y_2, x_2) \\ &\leq f(y_1) + f(x_2) + f(y_2) + f(x_1), \end{aligned}$$

and so $d_{\mathcal{S},\alpha}(y_1, x_1) = f(y_1) + f(x_1)$ and $d_{\mathcal{S},\alpha}(y_2, x_2) = f(y_2) + f(x_2)$. Hence, $\{x_1, y_1\}$ is an edge of $K(f)$ which is a contradiction.

(iii): Suppose $\psi \in [\phi]$ and let $\{x, y\}$ be an edge of $K(\kappa(\phi))$. By Proposition 3.2, Corollary 3.3(ii), and Theorem 4.3 $\phi_x, \gamma^x, \phi, \gamma^y, \phi_y$ and $\phi_x, \gamma^x, \psi, \gamma^y, \phi_y$ are geodesics in $B(\mathcal{S}, \alpha)$. Thus, since γ^x and γ^y are gates in C, for x and y respectively,

$$d_{\mathcal{S},\alpha}(x, y) = \kappa(\phi)(x) + \kappa(\phi)(y) = \kappa(\psi)(x) + \kappa(\psi)(y).$$

Hence, $\{x, y\}$ is an edge of $K(\kappa(\psi))$. Thus, by (TS1), $\kappa(\psi) \in [\kappa(\phi)]$.

Conversely, suppose $\kappa(\psi) \in [\kappa(\phi)]$. We can assume $\mathcal{S}(\phi) \neq \mathcal{S}$ since otherwise $supp(\psi) \subseteq \mathcal{U}(\mathcal{S}) = supp(\phi)$ and so $\psi \in [\phi]$. We first claim that if $S \in \mathcal{S} - \mathcal{S}(\phi)$, then there exist elements $x, y \in X$ with $S(x) = S(y)$ and γ^x the antipode of γ^y in $[\phi]$. Indeed, suppose $S \in \mathcal{S} - \mathcal{S}(\phi)$. By (B4), $\mathcal{S}(\phi)$ is a maximal incompatible split system in $\mathcal{S}$, and so there exists some $S' \in \mathcal{S}(\phi)$ with S' and S compatible. Hence there exists some $x \in X$ with $S(x) \cup S'(x) = X$. Since $[\phi]$ is antipodal X-gated by Theorem 4.3, there exists some $y \in X$ with γ^x is the antipode of γ^y in $[\phi]$. By Proposition 3.2, $y \notin S'(x)$ and so $y \in S(x)$. Hence, $S(x) = S(y)$, which completes the proof of the first claim.

We now claim that $\phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. Suppose $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$. Put $S_0 = S_A$. Then, by the claim just above, there exist elements $x, y \in X$ with $S_0(x) = S_0(y)$ and γ^x the antipode of γ^y in $[\phi]$. Hence, by Proposition 3.2 and Corollary 3.3(i), $\phi(S_0(x)) = 0$, and, by Lemma 4.1, $\{x, y\}$ is an edge of $K(\kappa(\phi)) \subseteq K(\kappa(\psi))$. Thus, $d_1(\phi_x, \psi) + d_1(\psi, \phi_y) = d_{\mathcal{S},\alpha}(x, y)$. Since for all $S \in \mathcal{S}$

$$\alpha_S \delta_S(x, y) \le \sum_{A \in S} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|,$$

it follows that $0 = \alpha_{S_0}\delta_{S_0}(x, y) = \sum_{A \in S_0} |\phi_x(A) - \psi(A)| + |\psi(A) - \phi_y(A)|$. Thus, $\phi_x(A) = \psi(A)$, for all $A \in S_0$, and so $\psi(S_0(x)) = 0$. In particular, it follows that $\phi(A) = \psi(A)$ holds for all $A \in \mathcal{U}(\mathcal{S} - \mathcal{S}(\phi))$ which concludes the proof of the claim. Using (B1), it is now straight-forward to conclude that $\psi \in [\phi]$. □

In view of the last theorem it follows that the map $\kappa' = \kappa'_{\mathcal{S},\alpha}$ defined by taking any maximal cell C in $B(\mathcal{S}, \alpha)$ to the cell $[\kappa(\phi)]$, where ϕ is any generator of C, is a well-defined map from the set of maximal cells of $B(\mathcal{S}, \alpha)$ to the set of maximal cells of $T(d_{\mathcal{S},\alpha})$. Moreover, we have

Corollary 5.2. *If $(\mathcal{S}, \alpha)$ is a weighted split system on X with $\mathcal{S}$ Teutoburgan, then the map κ' defined above is injective.*

Proof. Suppose that C and C' are maximal cells in $B(\mathcal{S}, \alpha)$ with $\kappa'(C) = \kappa'(C')$. Let ϕ and ϕ' be generators for C and C', respectively. Then $[\kappa(\phi)] = [\kappa(\phi')]$. Hence $\kappa(\phi) \in [\kappa(\phi')]$ and so $\phi \in [\phi']$ by Theorem 5.1(iii). Thus $[\phi] \subseteq [\phi']$ by (TS1). Interchanging the roles of ϕ and ϕ' yields $[\phi'] \subseteq [\phi]$. Therefore $C = [\phi] = [\phi'] = C'$. Hence κ' is injective. □

6. Totally split-decomposable metrics

For $(\mathcal{S}, \alpha)$ a weighted split system on X, by the main result of [9] $\kappa(B(\mathcal{S}, \alpha)) = T(d_{\mathcal{S},\alpha})$ if and only if $\mathcal{S}$ is weakly compatible. We now use this fact to prove that in case $\mathcal{S}$ is a weakly compatible split system, the map κ' defined at the end of the last section is a bijection.

Theorem 6.1. *Let $(\mathcal{S}, \alpha)$ be a weighted split system on X. If $\mathcal{S}$ is weakly compatible, then the map κ' is a bijection between the set of maximal cells of $B(\mathcal{S}, \alpha)$ and the set of maximal cells of $T(d_{\mathcal{S},\alpha})$.*

Proof. Since any weakly compatible split system is Teutoburgan, by Corollary 5.2 it follows that the map κ' is injective. Hence it suffices to prove that κ' is surjective.

To this end, suppose that Z is a maximal cell in $T(d_{\mathcal{S},\alpha})$. Let h be any generator of Z. Since $\mathcal{S}$ is weakly compatible κ maps $B(\mathcal{S},\alpha)$ onto $T(d_{\mathcal{S},\alpha})$ [9]. Hence, there must be some $\psi \in B(\mathcal{S},\alpha)$ with $\kappa(\psi) = h$. Suppose C is a maximal cell in $B(\mathcal{S},\alpha)$ which contains ψ, and let ϕ be a generator of C. Since $\mathcal{S}$ is Teutoburgan, $[\kappa(\phi)]$ is a maximal cell in $T(d_{\mathcal{S},\alpha})$ by Theorem 5.1(ii). So $h = \kappa(\psi) \in [\kappa(\phi)]$ by Theorem 5.1(iii), and thus, by (TS1), $Z = [h] \subseteq [\kappa(\phi)]$. But Z is maximal, and so $Z = [\kappa(\phi)]$. Thus κ' is surjective. □

We conclude this section by giving some new characterisations of weakly compatible split systems (see [9] for some further characterisations). Given a metric d on a finite set Y, define the *underlying graph* $UG(Y,d)$ to be the graph with vertex set Y and edge set consisting of those subsets $\{x, y\} \subseteq Y$ for which there is no $z \in Y$ distinct from x and y with $d(x, y) = d(x, z) + d(z, y)$. In addition, define a split system $\mathcal{S} \subseteq \mathcal{S}(X)$ to be *3-cube-free* if for all 3-subsets $\{S_1, S_2, S_3\} \subseteq \mathcal{S}$ there exists $A_k \in S_k$ for $k = 1, 2, 3$ with $A_1 \cap A_2 \cap A_3 = \emptyset$.

Theorem 6.2. *Suppose that $\mathcal{S} \subseteq \mathcal{S}(X)$ is a Teutoburgan split system. Then the following statements are equivalent.*

(i) *$\mathcal{S}$ is weakly compatible.*
(ii) *$\mathcal{S}$ is 3-cube-free.*
(iii) *for every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S},\alpha)$ with $\dim(C) = 3$, then $|\Gamma(C)| \leq 6$.*
(iii′) *for some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S},\alpha)$ with $\dim(C) = 3$, then $|\Gamma(C)| \leq 6$.*
(iv) *for every weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S},\alpha)$ with $\dim(C) \neq 0$, then $d_1|_{\Gamma(C)}$ is totally split-decomposable.*
(iv′) *for some weighting $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$, if C is a cell in $B(\mathcal{S},\alpha)$ with $\dim(C) \neq 0$, then $d_1|_{\Gamma(C)}$ is totally split-decomposable.*

Proof. Clearly (iii) ⇒ (iii′) and (iv) ⇒ (iv′).

(i) ⇒ (iv): Suppose $\mathcal{S}$ is weakly compatible, $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, C is a cell of $B(\mathcal{S},\alpha)$ with $\dim(C) > 0$, and ϕ is a generator of C. Then, by Lemma 3.1(iii),

$$d_1(\gamma^x, \gamma^y) = \sum_{S\in\mathcal{S}(\phi)} \alpha_S \delta_S(x, y)$$

for all $x, y \in X$. Since $\mathcal{S}(\phi) \subseteq \mathcal{S}$ and $\mathcal{S}$ is weakly compatible, $\mathcal{S}(\phi)$ is weakly compatible, and hence $d_1|_{\Gamma(C)}$ is totally split-decomposable.

(iv) ⇒ (iii): Suppose $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting, and that there is some 3-dimensional cell C in $B(\mathcal{S},\alpha)$ with $|\Gamma(C)| \geq 7$. By Theorem 4.3 C is antipodal X-gated, and, since C has eight vertices, $|\Gamma(C)| = 8$. Suppose ϕ is a generator of C and $x_0 \in X$. Then, by (B5), $|\mathcal{S}(\phi)| = 3$. Put $\mathcal{S}(\phi) = \{S_1, S_2, S_3\}$. Since each vertex of C is a gate, $\bigcap_{i=1}^{3} A_i \neq \emptyset$ for all $A_i \in S_i$, $i = 1, 2, 3$, and so we can choose some $x_i \in \overline{S_i}(x_0) \cap S_j(x_0) \cap S_k(x_0)$ with $\{i, j, k\} = \{1, 2, 3\}$. But then, by Lemma 3.1(iii),

$$d_1(\gamma^{x_0}, \gamma^{x_i}) = \sum_{S\in\mathcal{S}(\phi)} \alpha_S \delta_S(x_0, x_i) = \alpha_{S_i}$$

holds for $i = 1, 2, 3$. It follows that γ^{x_0} is a vertex in $UG(\Gamma(C), d_1|_{\Gamma(C)})$ with degree 3. But, since we are assuming that $d_1|_{\Gamma(C)}$ is totally split-decomposable, it follows by [16, Theorem 1.2] that $UG(\Gamma(C), d_1|_{\Gamma(C)})$ is an 8-cycle. This is a contradiction.
(iv$'$) $\Rightarrow$ (iii$'$): This can be proven using similar arguments to (iv) $\Rightarrow$ (iii).
(iii) $\Rightarrow$ (ii): This follows in a straight-forward manner from the definition of γ^z, $z \in X$.
(ii) $\Rightarrow$ (i): Suppose that $\alpha : \mathcal{S} \to \mathbb{R}^{>0}$ is a weighting and that $\mathcal{S}$ is not weakly compatible. Then there exist distinct splits $S_1, S_2, S_3 \in \mathcal{S}$ and distinct elements $x_0, x_1, x_2, x_3 \in X$ so that (1) holds. Note that $\mathcal{S}' = \{S_1, S_2, S_3\}$ is incompatible. Hence, $B(\mathcal{S}', \alpha|_{\mathcal{S}'}) = H(\mathcal{S}', \alpha|_{\mathcal{S}'})$ by [8, Proposition 3.3], and so there must exist some $\phi' \in B(\mathcal{S}', \alpha|_{\mathcal{S}'})$ with $\mathcal{S}(\phi') = \mathcal{S}'$. By (B2) there exists some $\phi \in B(\mathcal{S}, \alpha)$ with $\phi|_{\mathcal{S}'} = \phi'$. Without loss of generality, we may assume that $\mathcal{S}(\phi) = \mathcal{S}(\phi') = \mathcal{S}'$. Let $C = [\phi]$. Since $\mathcal{S}$ is Teutoburgan, Theorem 4.3 implies that C is antipodal X-gated and, by (B5), $\dim(C) = |\mathcal{S}(\phi)| = 3$. Now by (1), for all $i = 1, 2, 3$ and all $k, l \in \{1, 2, 3\} - i$ distinct, $x_i \in S_i(x_0) \cap \overline{S_k}(x_0) \cap \overline{S_l}(x_0)$ and so $\gamma^{x_0}, \gamma^{x_1}, \gamma^{x_2}, \gamma^{x_3}$ are all distinct gates in C and, by Proposition 3.2, for all $i, j \in \{0, 1, 2, 3\}$ the antipode of γ^{x_i} in C is not γ^{x_j}. Hence, $|\Gamma(C)| = 8$. But then, by the definition of γ^{x_j}, $j = 0, 1, 2, 3$, $\bigcap_{i=1,2,3} A_i \neq \emptyset$ where $A_i \in S_i$, $i = 1, 2, 3$. It follows that $\mathcal{S}$ is not 3-cube-free. $\square$

7. Cell-decomposability

Before proving our main result, we recall from [16] the definition of cell-decomposable metrics and a result about such metrics that will be key in our proof.

Suppose that d is a metric on X. Given a cell C of $T(d)$ and some $x \in X$, we call a (necessarily unique) element $g \in C$ a *gate in C for x* if, for all $h \in C$,

$$d_\infty(h_x, h) = d_\infty(h_x, g) + d_\infty(g, h).$$

We say that C is *X-gated* if there is a gate in C for each $x \in X$. In case every cell in $T(d)$ is X-gated we call d *cell-decomposable*.

For d a metric and C a cell of $T(d)$, we let $G(C)$ be the set of gates in C for all the elements in X. We shall use the following restatement of [16, Theorem 1.1]:

- Suppose that d is a cell-decomposable metric. Then the metric $d_\infty|_{G(C)}$ is antipodal, and the map $\chi = \chi_C : C \to T(d_\infty|_{G(C)})$ defined, for all $f \in C$ and all $a \in G(C)$ and $x \in X$ with $a = f^x$, by $\chi(f)(a) = f(x) - f^x(x)$ is a bijective isometry that induces a polytope isomorphism between C and $T(d_\infty|_{G(C)})$.

In [16] we conjectured that a metric is totally split-decomposable if and only if it is cell-decomposable. We now prove that the "only if" direction of this conjecture holds.

Theorem 7.1. *If d is a totally split-decomposable metric, then d is cell-decomposable.*

Proof. Suppose that d is totally split-decomposable. Let $(\mathcal{S}, \alpha)$ be the unique weighted split system on X with $\mathcal{S}$ weakly compatible and $d = d_{\mathcal{S},\alpha}$. Note that since $\mathcal{S}$ is weakly compatible it is Teutoburgan.

Now, let Z be a maximal cell of $T(d)$. Suppose that C is the maximal cell in $B(\mathcal{S}, \alpha)$ with $\kappa'(C) = Z$, which exists by Theorem 6.1.

Claim 1: Z is X-gated.

Let x be any element of X. We will show that $\kappa(\gamma_C^x)$ is a gate for x in Z.

Suppose $f \in Z$ and let ϕ be a generator of C. Then, since κ is surjective, there must exist some $\psi \in B(\mathcal{S}, \alpha)$ with $\kappa(\psi) = f$. Since C is a maximal cell, $\psi \in C$ by Theorem 5.1(iii). Since $\mathcal{S}$ is Teutoburgan, by Theorem 4.3 and Corollary 3.3(ii) there must exist some $y \in X$ with $\phi_x, \gamma_C^x, \phi, \gamma_C^y, \phi_y$ a geodesic in $B(\mathcal{S}, \alpha)$. By Proposition 3.2, the fact that κ is a non-expanding map, and, by Theorem 4.3, it follows that $\kappa(\phi_x), \kappa(\gamma^x), f, \kappa(\gamma^y), \kappa(\phi_y)$ is a geodesic in $T(d)$. But then by (TS4)

$$d_\infty(\kappa(\phi_x), \kappa(\gamma^x)) + d_\infty(\kappa(\gamma^x), f) = d_\infty(\kappa(\phi_x), f) = d_\infty(h_x, f).$$

Hence $\kappa(\gamma_C^x)$ is a gate for x in Z. This completes the proof of Claim 1.

Put $d' := d_\infty|_{G(Z)}$. Note that by [16, Theorem 1.1] d' is antipodal.

Claim 2: d' is totally split-decomposable.

Since Z is X-gated and κ is a non-expanding map, κ induces an isometry between $(\Gamma(C), d_1|_{\Gamma(C)})$ and $(G(Z), d')$. Hence, since $\mathcal{S}$ is weakly compatible and so, by Theorem 6.2, $d_1|_{\Gamma(C)}$ is totally split-decomposable we also have that d' is totally split-decomposable. This completes the proof of Claim 2.

Since d' is antipodal and totally split-decomposable, it immediately follows by [16, Theorem 1.2] that d' is cell-decomposable. The theorem now follows directly from:

Claim 3: Every cell in $T(d)$ is X-gated.

Let W be any cell of $T(d)$, and x be any element of X. Suppose that Z is any maximal cell in $T(d)$ containing W.

Since Z is X-gated by Claim 1, there is a gate g^x for x in Z. Let $\chi_Z : Z \to T(d')$ be the map given by [16, Theorem 1.1]. Since $d' = d_\infty|_{G(Z)}$ is cell decomposable, there is a gate for $\chi(g^x)$ in $\chi(W)$. Let p be the inverse image under χ of this gate. We will show that p is a gate for x in W from which the claim follows.

Let $f \in W$. Since χ is a bijective isometry

$$d_\infty(g^x, f) = d_\infty(g^x, p) + d_\infty(p, f),$$

and, since Z is X-gated,

$$d_\infty(x, f) = d_\infty(x, g^x) + d_\infty(g^x, f).$$

But, since $p \in Z$ and Z is X-gated,

$$d_\infty(x, p) = d_\infty(x, g^x) + d_\infty(g^x, p).$$

Hence, in view of these last three equalities, it immediately follows that $d_\infty(x, f) = d_\infty(x, p) + d_\infty(p, f)$. Thus, p is a gate for x in W. This concludes the proof of Claim 3. $\square$

Before proving our final result, we present a result that we will need concerning the tight-span of an antipodal metric. Recall that a polytope in $\mathbb{R}^n$ is *centrally symmetric* if each cell P contains a point c called the *centre* of P such that $c + x \in P$ if and only if $c - x \in P$, for $x \in \mathbb{R}^n$.

Lemma 7.2. *Suppose that d is an antipodal metric on a finite set X. Then $T(d)$ is a centrally symmetric polytope.*

Proof. By [15, Theorem 4.2], $T(d)$ is a polytope in $\mathbb{R}^X$. To see that $T(d)$ is centrally symmetric, we have to show that $T(d)$ contains a centre, that is, some map $f \in T(d)$ with $f+g \in T(d)$ if and only if $f-g \in T(d)$, for all $g \in \mathbb{R}^X$. Consider the map

$$f : X \to \mathbb{R}^{\geq 0} : y \mapsto d(y, \overline{y})/2.$$

By [15, Lemma 3.1], $d(x, \overline{y}) = d(y, \overline{x})$ for all $x, y \in X$. Hence, $f \in P(d)$. Moreover, by [15, Lemma 4.1], in which it is shown that a map $h \in P(d)$ is contained in $T(d)$ if and only if, for all $x \in X$, $h(x) + h(\overline{x}) = d(x, \overline{x})$ it immediately follows that $f \in T(d)$.

Now suppose $g \in \mathbb{R}^X$. We will show that if $f+g \in T(d)$, then $f-g \in T(d)$, which will complete the proof of the lemma. Suppose that $f+g \in T(d)$. We start with showing $f-g \in P(d)$. To this end, suppose $x, y \in X$. Then, by the definition of f, [15, Lemma 3.1], and [15, Lemma 4.1],

$$\begin{aligned}
&f(x) - g(x) + f(y) - g(y) \\
&\quad = f(x) - g(x) + f(x) + g(x) + f(\overline{x}) + g(\overline{x}) - d(x, \overline{x}) \\
&\qquad + f(y) - g(y) + f(y) + g(y) + f(\overline{y}) + g(\overline{y}) - d(y, \overline{y}) \\
&\quad = 2(f(x) + f(y)) + (g+f)(\overline{x}) + (g+f)(\overline{y}) - d(x, \overline{x}) - d(y, \overline{y}) \\
&\quad = (g+f)(\overline{x}) + (g+f)(\overline{y}) \geq d(\overline{x}, \overline{y}) = d(x, y),
\end{aligned}$$

and so $f-g$ is an element in $P(d)$. Using [15, Lemma 4.1] again and the fact that f and $f+g$ are elements of $T(d)$ it is easily seen that $f-g \in T(d)$. □

Recall that a *zonotope* is a polytope in $\mathbb{R}^n$ all of whose cells are centrally symmetric [21, p. 201]. Note that even though the tight-span of an antipodal metric is centrally symmetric, it is not necessarily a zonotope (for example, the tight-span of the graph metric associated to the 3-cube contains 2-dimensional cells that are polytope isomorphic to triangles [14]).

We now prove our final result.

Corollary 7.3. *Suppose that d is a totally split-decomposable metric. Then every cell in $T(d)$ is a zonotope that is polytope isomorphic either to a hypercube or to a rhombic dodecahedron.*

Proof. We first show that every cell Z in $T(d)$ is centrally symmetric from which it immediately follows that Z is a zonotope. Suppose that Z is any cell of $T(d)$. Denote the metric $d_\infty|_{G(Z)}$ by d'. By Theorem 7.1, d is cell-decomposable. Hence Z is X-gated and so, by [16, Theorem 1.1], d' is antipodal. Thus by Lemma 7.2, $T(d')$ is a centrally symmetric polytope.

Now let $\chi_Z : Z \to T(d')$ be the map given by [16, Theorem 1.1]. Suppose $f \in Z$ with $\chi(f)$ the centre of $T(d')$. We claim that f is a centre for Z.

Suppose $g \in \mathbb{R}^X$ with $f+g \in Z$. Then, since χ is a bijection, $\chi(f+g) \in T(d')$ and so $\chi(f) + (\chi(f+g) - \chi(f)) \in T(d')$. Since $\chi(f)$ is a centre for $T(d')$, we also have $\chi(f) - (\chi(f-g) - \chi(f)) \in T(d')$ and so $2\chi(f) - \chi(f-g) \in T(d')$. Suppose $a \in G(Z)$ and $x \in X$ with $a = f^x$. Then, by definition of χ,

$$\begin{aligned}
2\chi(f)(a) - \chi(f-g)(a) &= 2f(x) - 2f^x(x) - f(x) - g(x) + f^x(x) \\
&= \chi(f-g)(a).
\end{aligned}$$

Hence, $\chi(f-g) \in T(d')$ and so $f-g \in Z$. Thus, f is a centre for Z, as claimed.

To complete the proof of the corollary we must show that every cell in $T(d)$ is polytope isomorphic either to a hypercube or to the rhombic dodecahedron. It suffices to show that this holds for every maximal cell of $T(d)$.

Suppose that Z is a maximal cell. Then by Claim 2 in the proof of Theorem 7.1, the metric $d_\infty|_{G(Z)}$ is totally split-decomposable and, by [16, Theorem 1.1], it is antipodal. It immediately follows by [16, Theorem 1.2] that $T(d_\infty|_{G(Z)})$ is polytope isomorphic to either a hypercube or a rhombic dodecahedron. The proof is completed by applying [16, Theorem 1.1]. □

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References

[1] J.-P. Allain, Y. Dong, A.-M. Vandamme, V. Moulton, M. Salmei, Evolutionary rate and genetic drift of hepatitis C virus are not correlated with the host immune response: studies of infected donor-recipient clusters, Journal of Virology 74 (2000) 2541–2549.
[2] H.J. Bandelt, A. Dress, A canonical decomposition theory for metrics on a finite set, Advances in Mathematics 92 (1992) 47–105.
[3] M. Chorbak, L. Lamore, Generosity helps or an 11-competitive algorithm for three servers, Journal of Algorithms 16 (1994) 234–263.
[4] M. Develin, B. Sturmfels, Tropical convexity, Documenta Mathematica 9 (2004) 1–27.
[5] M. Develin, B. Sturmfels, Erratum for tropical convexity, Documenta Mathematica 9 (2004) 205–206.
[6] M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer-Verlag Telos, 1997.
[7] A. Dress, Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups: a note on combinatorial properties of metric spaces, Advances in Mathematics 53 (1984) 321–402.
[8] A. Dress, K.T. Huber, V. Moulton, Some variations on a theme by Buneman, Annals of Combinatorics 1 (1997) 339–352.
[9] A. Dress, K.T. Huber, V. Moulton, A comparison between two distinct models in projective cluster theory: the median and the tight-span construction, Annals of Combinatorics 2 (1998) 299–311.
[10] A. Dress, K.T. Huber, V. Moulton, Totally split-decomposable metrics of combinatorial dimension two, Annals of Combinatorics 5 (2001) 97–110.
[11] A. Dress, K.T. Huber, V. Moulton, Antipodal metrics and split systems, European Journal of Combinatorics 23 (2002) 187–200.
[12] A. Dress, K.T. Huber, V. Moulton, An explicit computation of the injective hull of certain finite metric spaces in terms of their associated Buneman complex, Advances in Mathematics 168 (2002) 1–28.
[13] A. Dress, D. Huson, V. Moulton, Analyzing and visualizing distance data using SplitsTree, Discrete Applied Mathematics 71 (1996) 95–110.
[14] O. Goodmann, V. Moulton, On the tight-span of an antipodal graph, Discrete Mathematics 218 (2000) 73–96.
[15] K.T. Huber, J. Koolen, V. Moulton, The tight-span of an antipodal metric space: part I—combinatorial properties, Discrete Mathematics (in press).
[16] K.T. Huber, J. Koolen, V. Moulton, The tight-span of an antipodal metric space: part II—geometrical properties, Discrete and Computational Geometry 31 (4) (2004) 567–586.

[17] D. Huson, SplitsTree: a program for analyzing and visualizing evolutionary data, Bioinformatics 14 (1) (1998) 68–73.
[18] J. Isbell, Six theorems about metric spaces, Commentarii Mathematici Helvetici 39 (1964) 65–74.
[19] V. Klee, P. Kleinschmidt, Convex polytopes and related complexes, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Part I, Elsevier Science, North-Holland, 1999.
[20] P. Lockhart, P. McLenachan, D. Havell, D. Glenny, D. Huson, U. Jensen, Phylogeny, dispersal and radiation of New Zealand alpine buttercups: molecular evidence under split decomposition, Annals of the Missouri Botanical Garden 88 (2001) 458–477.
[21] G. Ziegler, Lectures on Polytopes, Springer-Verlag, 1995.